Please check the examination details below before entering your candidate information

Candidate surname	Other names

Centre Number

Candidate Number

Pearson Edexcel Level 3 GCE

Time 1 hour 30 minutes

Paper reference **9FM0/4A**

Further Mathematics

Advanced

PAPER 4A: Further Pure Mathematics 2

You must have:
Mathematical Formulae and Statistical Tables (Green), calculator

Total Marks

Candidates may use any calculator permitted by Pearson regulations. Calculators must not have the facility for symbolic algebraic manipulation, differentiation and integration, or have retrievable mathematical formulae stored in them.

Instructions

- Use **black** ink or ball-point pen.
- If pencil is used for diagrams/sketches/graphs it must be dark (HB or B).
- **Fill in the boxes** at the top of this page with your name, centre number and candidate number.
- Answer **all** questions and ensure that your answers to parts of questions are clearly labelled.
- Answer the questions in the spaces provided – *there may be more space than you need.*
- You should show sufficient working to make your methods clear. Answers without working may not gain full credit.
- Inexact answers should be given to three significant figures unless otherwise stated.

Information

- A booklet 'Mathematical Formulae and Statistical Tables' is provided.
- There are 10 questions in this question paper. The total mark for this paper is 75.
- The marks for **each** question are shown in brackets – *use this as a guide as to how much time to spend on each question.*

Advice

- Read each question carefully before you start to answer it.
- Try to answer every question.
- Check your answers if you have time at the end.

Turn over ▶

2.1: The Axioms For A Group | 2.2: Cayley Tables & Finite Groups | 2.3: Order & Subgroups

1. The group S_4 is the set of all possible permutations that can be performed on the four numbers 1, 2, 3 and 4, under the operation of composition.

For the group S_4

(a) write down the identity element,

(1)

(b) write down the inverse of the element a, where

$$a = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 2 & 1 \end{pmatrix}$$

(1)

(c) demonstrate that the operation of composition is associative using the following elements

$$a = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 2 & 1 \end{pmatrix} \quad b = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 1 \end{pmatrix} \quad \text{and } c = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix}$$

(2)

(d) Explain why it is possible for the group S_4 to have a subgroup of order 4
You do not need to find such a subgroup.

(2)

a. $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}$ // maps to same elements

b. mapping of a: $1 \to 3$
$2 \to 4$
$3 \to 2$
$4 \to 1$

must now map backwards:
$3 \to 1$
$4 \to 2$
$2 \to 3$
$1 \to 4$

top row written in order:
$a^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 1 & 2 \end{pmatrix}$ //

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Question 1 continued

c. if associative: $a * (b * c) = (a * b) * c$

$a * b = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix}$ | $b * c = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 3 \end{pmatrix}$

$(a * b) * c = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$ | $a * (b * c) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$

$(a * b) * c = a * (b * c)$

$\therefore$ Group S_4 is associative //

d. order of group S_4 is $4!$ (24)

since $4 | 24$, it is possible for a subgroup to have order 4 (Lagrange's theorem)

part (c) explained how to get composition:

$a * b$

$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 2 & 1 \end{pmatrix} * \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 1 \end{pmatrix}$

Start with the element on right (b), and write down its mapping:

$1 \to 2$

$2 \to 4$

$3 \to 3$

$4 \to 1$

now check what these no.s $\{2, 4, 3, 1\}$ map to in A

$1 \to 2 \to 4$

$2 \to 4 \to 1$ in final permutation

$3 \to 3 \to 2$ leave out middle row

$4 \to 1 \to 3$

$a * b = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix}$

(Total for Question 1 is 6 marks)

5.1: Eigenvalues & Eigenvectors
5.3: The Cayley-Hamilton Theorem

2. Matrix **M** is given by

$$\mathbf{M} = \begin{pmatrix} 1 & 0 & a \\ -3 & b & 1 \\ 0 & 1 & a \end{pmatrix}$$

where a and b are integers, such that $a < b$

Given that the characteristic equation for **M** is

$$\lambda^3 - 7\lambda^2 + 13\lambda + c = 0$$

where c is a constant,

(a) determine the values of a, b and c.

(5)

(b) Hence, using the Cayley–Hamilton theorem, determine the matrix $\mathbf{M}^{-1}$

(3)

a. characteristic eq^n: $\det(M - \lambda I) = 0$

$$M - \lambda I = \begin{pmatrix} 1 & 0 & a \\ -3 & b & 1 \\ 0 & 1 & a \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$M - \lambda I = \begin{pmatrix} 1 & 0 & a \\ -3 & b & 1 \\ 0 & 1 & a \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}$$

$$M - \lambda I = \begin{pmatrix} 1-\lambda & 0 & a \\ -3 & b-\lambda & 1 \\ 0 & 1 & a-\lambda \end{pmatrix}$$

$\det(M - \lambda I) = 0$

$$\det\left[\begin{pmatrix} 1-\lambda & 0 & a \\ -3 & b-\lambda & 1 \\ 0 & 1 & a-\lambda \end{pmatrix}\right] = 0$$

$$(1-\lambda)\begin{vmatrix} b-\lambda & 1 \\ 1 & a-\lambda \end{vmatrix} - \cancel{(0)\begin{vmatrix} -3 & 1 \\ 0 & a-\lambda \end{vmatrix}} + (a)\begin{vmatrix} -3 & b-\lambda \\ 0 & 1 \end{vmatrix} = 0$$

Question 2 continued

$(1-\lambda)\left[(b-\lambda)(a-\lambda)-(1)(1)\right]+(a)\left[(-3)(1)-\cancel{(0)(b-\lambda)}\right]=0$

$(1-\lambda)\left[ab-a\lambda-b\lambda+\lambda^2-1\right]+a\left[-3\right]=0$

$ab-a\lambda-b\lambda+\lambda^2-1-ab\lambda+a\lambda^2+b\lambda^2-\lambda^3+\lambda-3a=0$

$-\lambda^3+(1+a+b)\lambda^2-(a+b+ab-1)\lambda-(3a+1-ab)=0$

$\times -1$

$\lambda^3-(1+a+b)\lambda^2+(a+b+ab-1)\lambda+(3a+1-ab)=0$

$1+a+b=7$
$a+b=6$

$a+b+ab-1=13$
$a+b+ab=14$
$a+b+ab=14$
$6+ab=14$
$ab=8$
$a=8/b$

equate coefficients

$\frac{8}{b}+b=6$ $\quad \times b$
$8+b^2=6b$
$b^2-6b+8=0$
$(b-4)(b-2)=0$
$b=2 \quad b=4$

if: $b=2 \quad a=8/2=4 \qquad a=4, b=2$
$\quad b=4 \quad a=8/4=2 \qquad a=2, b=4$

Since $a<b$, $a=2$ and $b=4$
$c=3a+1-ab$
$c=3(2)+1-(2)(4)$

$c=-1$ //

Question 2 continued

b. use Cayley hamilton theorem:

characteristic eq^n $\lambda^3 - 7\lambda^2 + 13\lambda - 1 = 0$ can be written as $M^3 - 7M^2 + 13M - I = 0$

$M^3 - 7M^2 + 13M - I = 0$) $\div M$

$M^2 - 7M + 13I - M^{-1} = 0$

$M^{-1} = M^2 - 7M + 13I$

$M^2 = \begin{pmatrix} 1 & 2 & 6 \\ -15 & 17 & 0 \\ -3 & 6 & 5 \end{pmatrix}$ can get directly from calculator

$M^{-1} = \begin{pmatrix} 1 & 2 & 6 \\ -15 & 17 & 0 \\ -3 & 6 & 5 \end{pmatrix} - 7\begin{pmatrix} 1 & 0 & 2 \\ -3 & 4 & 1 \\ 0 & 1 & 2 \end{pmatrix} + 13\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$M^{-1} = \begin{pmatrix} 1 & 2 & 6 \\ -15 & 17 & 0 \\ -3 & 6 & 5 \end{pmatrix} - \begin{pmatrix} 7 & 0 & 14 \\ -21 & 28 & 7 \\ 0 & 7 & 14 \end{pmatrix} + \begin{pmatrix} 13 & 0 & 0 \\ 0 & 13 & 0 \\ 0 & 0 & 13 \end{pmatrix}$

$\boxed{M^{-1} = \begin{pmatrix} 7 & 2 & -8 \\ 6 & 2 & -7 \\ -3 & -1 & 4 \end{pmatrix}}$

4.1: Forming Recurrence Relations   4.4: Proving Closed Forms

3.

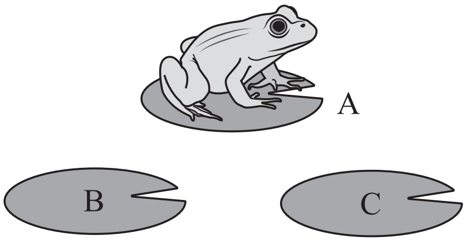

Figure 1

There are three lily pads on a pond. A frog hops repeatedly from one lily pad to another.

The frog starts on lily pad A, as shown in Figure 1.

In a model, the frog hops from its position on one lily pad to either of the other two lily pads with equal probability.

Let p_n be the probability that the frog is on lily pad A after n hops.

(a) Explain, with reference to the model, why $p_1 = 0$

(1)

The probability p_n satisfies the recurrence relation

$$p_{n+1} = \frac{1}{2}(1 - p_n) \qquad n \geqslant 1 \quad \text{where } p_1 = 0$$

(b) Prove by induction that, for $n \geqslant 1$

$$p_n = \frac{2}{3}\left(-\frac{1}{2}\right)^n + \frac{1}{3}$$

(6)

(c) Use the result in part (b) to explain why, in the long term, the probability that the frog is on lily pad A is $\frac{1}{3}$

(1)

a. Has to hop to different lilypad to A//

b. Basis step:

when $n=1$, $P_1 = \frac{2}{3}\left(-\frac{1}{2}\right)^1 + \frac{1}{3}$

$P_1 = 0$

Assumption step:

assume $n=k$, $P_k = \frac{2}{3}\left(-\frac{1}{2}\right)^k + \frac{1}{3}$

P 6 5 5 0 6 A 0 8 3 2

Question 3 continued

Inductive step:

$P_{k+1} = \frac{1}{2}(1 - P_k)$

$P_{k+1} = \frac{1}{2}\left(1 - \left[\frac{2}{3}\left(-\frac{1}{2}\right)^k + \frac{1}{3}\right]\right)$

$P_{k+1} = \frac{1}{2}\left(1 - \frac{2}{3}\left(-\frac{1}{2}\right)^k - \frac{1}{3}\right)$

$P_{k+1} = \frac{1}{2}\left(\frac{2}{3} - \frac{2}{3}\left(-\frac{1}{2}\right)^k\right)$

$P_{k+1} = \frac{1}{3} + \frac{2}{3}\left(-\frac{1}{2}\right)^k\left(-\frac{1}{2}\right)$

$P_{k+1} = \frac{1}{3} + \frac{2}{3}\left(-\frac{1}{2}\right)^{k+1}$ // (Result holds for $n = k+1$)

Conclusion step:

If true for $n=k$, then it is true for $n=k+1$, and as it is true for $n=1$, the statement is true for all n //

c. as $n \to \infty$, $\left(-\frac{1}{2}\right)^n \to 0$

$\therefore P_n = \frac{2}{3}\left(-\frac{1}{2}\right)^n + \frac{1}{3} \to 1/3$

$1/3$ //

4. (a) Use the Euclidean algorithm to show that 124 and 17 are relatively prime (coprime).
(2)

(b) Hence solve the equation

$$124x + 17y = 10$$

(3)

(c) Solve the congruence equation

$$124x \equiv 6 \mod 17$$

(2)

a. gcd (124, 17)

$124 = 7(17) + 5$

$17 = 3(5) + 2$

$5 = 2(2) + 1$

$2 = 2(1)$

$\therefore$ gcd (124, 17) = 1 so 124 and 17 are coprime

b. $124 = 7(17) + 5 \rightarrow 5 = 124 - 7(17)$

$17 = 3(5) + 2 \rightarrow 2 = 17 - 3(5)$

$5 = 2(2) + 1 \rightarrow 1 = 5 - 2(2)$

$2 = 2(1)$

from part (a) algorithm but now make the subject the remainder

sub in 2

$1 = 5 - 2[17 - 3(5)]$ as $\{17 - 3(5)\}$ from line ②

$1 = 5 - 2(17) + 6(5)$

$1 = 7(5) - 2(7)$

$1 = 7[124 - 7(17)] - 2(7)$ sub in 5 as $\{124 - 7(17)\}$ from line ①

$1 = 7(124) - 49(17) - 2(7)$

$1 = 7(124) - 51(17)$

$\therefore 124(7) + 17(-51) = 1$ $\times 10$

$124(70) + 17(-50) = 10$

$\therefore x = 70, \ y = -50$

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Question 4 continued

c. $124a + 17b = 1$

$124a \equiv 1 \mod 17$

$124(7) \equiv 1 \mod 17$

$\therefore$ multiplicative inverse : 7

$124x \equiv 6 \mod 17$

$7 \times 124x \equiv 7 \times 6 \mod 17$

$x \equiv 42 \mod 17$

$x \equiv 8 \mod 17$

$x \equiv 8 \mod 17$

(Total for Question 4 is 7 marks)

5. The locus of points z satisfies

$$|z + a\mathrm{i}| = 3|z - a|$$

where a is an integer.

The locus is a circle with its centre in the third quadrant and radius $\frac{3}{2}\sqrt{2}$

Determine

(a) the value of a,

(4)

(b) the coordinates of the centre of the circle.

(2)

a. USE ALGEBRAIC APPROACH OF EVALUATING LOCI:

$|z + ai| = 3|z - a|$

let $z = x + iy$

$|(x+iy) + ai| = 3|(x+iy) - a|$ — seperate into

$|(x) + (a+y)i| = 3|(x-a) + (y)i|$ — real & imaginary parts

$\sqrt{(x)^2 + (a+y)^2} = 3\sqrt{(x-a)^2 + (y)^2}$ — square both sides

$\left(\sqrt{(x)^2 + (a+y)^2}\right)^2 = (3)^2\left(\sqrt{(x-a)^2 + (y)^2}\right)^2$ — to remove sqrt $\sqrt{\ }$

$(x)^2 + (a+y)^2 = 9[(x-a)^2 + (y)^2]$

$x^2 + a^2 + 2ay + y^2 = 9[x^2 - 2ax + a^2 + y^2]$

$x^2 + a^2 + 2ay + y^2 = 9x^2 - 18ax + 9a^2 + 9y^2$

$8x^2 - 18ax + 8y^2 - 2ay + 8a^2 = 0$ — $\div 8$

$x^2 - \frac{18}{8}ax + y^2 - \frac{2}{8}ay + a^2 = 0$

$(x - \frac{9}{8}a)^2 - \frac{81}{64}a^2 + (y - \frac{1}{8}a)^2 - \frac{1}{64}a^2 + a^2 = 0$ — complete the square for x & y

$(x - \frac{9}{8}a)^2 + (y - \frac{1}{8}a)^2 - \frac{9}{32}a^2 = 0$

$(x - \frac{9}{8}a)^2 + (y - \frac{1}{8}a)^2 = \frac{9}{32}a^2$

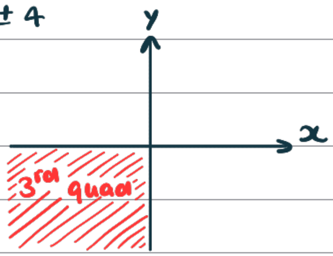
$r^2 = \frac{9}{32}a^2$

Q states $r = \frac{3}{2}\sqrt{2}$ $\therefore$ $r^2 = \left(\frac{3}{2}\sqrt{2}\right)^2 = \frac{9}{2}$

$\frac{9}{32}a^2 = \frac{9}{2}$

$a^2 = 16$

$a = \pm 4$



implies (x, y) from circle centre should be -v.e. values (both)

Question 5 continued

when $a=4$, $(x-\frac{9}{2})^2+(y-\frac{1}{2})^2=\frac{9}{2}$ centre: $(\frac{9}{2},\frac{1}{2})$

$a=-4$, $(x+\frac{9}{2})^2+(y+\frac{1}{2})^2=\frac{9}{2}$ centre: $(-\frac{9}{2},-\frac{1}{2})$ ← lies in 3rd quadrant

$a=-4$

b. $(-\frac{9}{2},-\frac{1}{2})$

use working above

(Total for Question 5 is 6 marks)

4.3: Solving 2nd Order Recurrence Relations

6. (a) Determine the general solution of the recurrence relation

$$u_n = 2u_{n-1} - u_{n-2} + 2^n \qquad n \geqslant 2$$

(4)

(b) Hence solve this recurrence relation given that $u_0 = 2u_1$ and $u_4 = 3u_2$

(2)

a. Associated homogenous recurrence relation:

$u_n = 2u_{n-1} - u_{n-2}$

$u_n - 2u_{n-1} + u_{n-2} = 0$

$r^2 - 2r + 1 = 0$

$(r-1)^2 = 0$

$r = 1$

complementary function (C.F.):

$u_n = (A+Bn)(1)^n$

Try the particular integral (P.I.): $u_n = \mu 2^n$

$u_{n-1} = \mu 2^{(n-1)} = \mu(2^n)(2^{-1}) = \frac{1}{2}\mu 2^n$

$u_{n-2} = \mu 2^{(n-2)} = \mu(2^n)(2^{-2}) = \frac{1}{4}\mu 2^n$

$\mu 2^n = 2\left(\frac{1}{2}\mu 2^n\right) - \frac{1}{4}\mu 2^n + 2^n$

$\mu 2^n = \mu 2^n - \frac{1}{4}\mu 2^n + 2^n$

$\frac{1}{4}\mu 2^n = 1 2^n$

$\left.\begin{array}{l}\frac{1}{4}\mu = 1 \\ \mu = 4\end{array}\right\}$ equating coefficients

P.I: $4(2^n)$

gen sol^n: C.F. + P.I

$u_n = (A+Bn)(1)^n + 4(2^n)$

for any $n \in \mathbb{N}$, $(1)^n = 1$ $\therefore$ don't need to include

$u_n = (A+Bn) + 4(2^n)$ //

Question 6 continued

b. find u_0, u_1, u_2, u_4:

$n=0$ $u_0 = (A + B(0)) + 4(2^0) = A + 4$

$n=1$ $u_1 = (A + B(1)) + 4(2^1) = A + B + 8$

$n=2$ $u_2 = (A + B(2)) + 4(2^2) = A + 2B + 16$

$n=4$ $u_4 = (A + B(4)) + 4(2^4) = A + 4B + 64$

$u_0 = 2u_1$

$A + 4 = 2[A + B + 8]$

$A + 4 = 2A + 2B + 16$

$A + 2B = -12$ (1)

$u_4 = 3u_2$

$A + 4B + 64 = 3[A + 2B + 16]$

$A + 4B + 64 = 3A + 6B + 48$

$2A + 2B = 16$

$A + B = 8$ (2)

Solve (1) and (2) simultaneously:

$A + 2B = -12$

$A + B = 8$ $-$

$B = -20$

$A + (-20) = 8$

$A = 28$

$A = 28$, $B = -20$

$u_n = (28 - 20n) + 4(2^n)$

(Total for Question 6 is 6 marks)

7. (i) The polynomial $F(x)$ is a quartic such that

$$F(x) = px^4 + qx^3 + 2x^2 + rx + s$$

where p, q, r and s are distinct constants.

Determine the number of possible quartics given that

(a) the constants p, q, r and s belong to the set $\{-4, -2, 1, 3, 5\}$
5 elements in set **(1)**

(b) the constants p, q, r and s belong to the set $\{-4, -2, 0, 1, 3, 5\}$
6 elements in set **(1)**

(ii) A 3-digit positive integer $N = abc$ has the following properties

- N is divisible by 11
- the sum of the digits of N is even
- $N \equiv 8 \bmod 9$

(a) Use the first two properties to show that

$$a - b + c = 0$$

(2)

(b) Hence determine all possible integers N, showing all your working and reasoning.

(4)

ia. p we can choose 1 of the 5 options
q we can choose 1 of the other 4 options
r we can choose 1 of the other 3 options
s we can choose 1 of the other 2 options

$5 \times 4 \times 3 \times 2$
$= 120$

b. p cannot be 0 $\therefore$ only 5 choices
q we can choose 1 of the other 5 options
r we can choose 1 of the other 4 options
s we can choose 1 of the other 3 options

$5 \times 5 \times 4 \times 3$
$= 300$

Question 7 continued

iia. $N = 100a + 10b + c$

#1: If divisible by 11, alternating sum of digits is divisible by 11

$\therefore a - b + c = 11k \qquad k \in \mathbb{Z}$

$1 \leq a \leq 9$
$0 \leq b \leq 9$
$0 \leq c \leq 9$

$(9) - (0) + (9) = 18$

$\therefore \max(a - b + c) = 18$
$k = 0$ or $k = 1$

$a - b + c = 0$ or 11

#2: $a + b + c = 2m \quad m \in \mathbb{Z}$

If we let $a - b + c = 11$
$a + b + c = 11 + 2b$ } +2b on both sides
$2m = 11 + 2b$
$\underbrace{}$ must be even from #2
However $11 + 2b \rightarrow$ odd + even = $\underline{\text{odd}}$ — we need even

$\therefore a - b + c \neq 11$
$\therefore a - b + c = 0$

b. $100a + 10b + c \equiv 8 \pmod 9$
$a + b + c \equiv 8 \pmod 9$

$a + b + c = 8, 17, 26$ (cannot be > 26 since $\max(a+b+c) = 27$ if $a, b, c = 9$)

Question 7 continued

$a+b+c = 8, 26$ (#2 states has to be even)

$a-b+c = 0$
$a+b+c = 8, 26$ (+)
$2a + 2c = 8, 26$
$a + c = 4, 13$

rearranging $a-b+c=0 \Rightarrow b = a+c$

$\therefore b = 4$ or 13
$b \neq 13$ has to be single digit
$\therefore b = 4$
$a+c = 4$
↳ go through all combinations that make this possible

a	b	c
1	4	3
2	4	2
3	4	1
4	4	0

(remember $a \neq 0$)

possible integers N: 143, 242, 341, 440

3.1: Loci on an Argand Diagram

8. The locus of points $z = x + iy$ that satisfy

$$\arg\left(\frac{z-8-5i}{z-2-5i}\right) = \frac{\pi}{3}$$ ← LOCUS EXPLAINED @ END OF Q

is an arc of a circle C.

(a) On an Argand diagram sketch the locus of z. **(2)**

(b) Explain why the centre of C has x coordinate 5 **(1)**

(c) Determine the radius of C. **(2)**

(d) Determine the y coordinate of the centre of C. **(2)**

a. $\arg\left(\frac{z-(8+5i)}{z-(2+5i)}\right) = \frac{\pi}{3}$

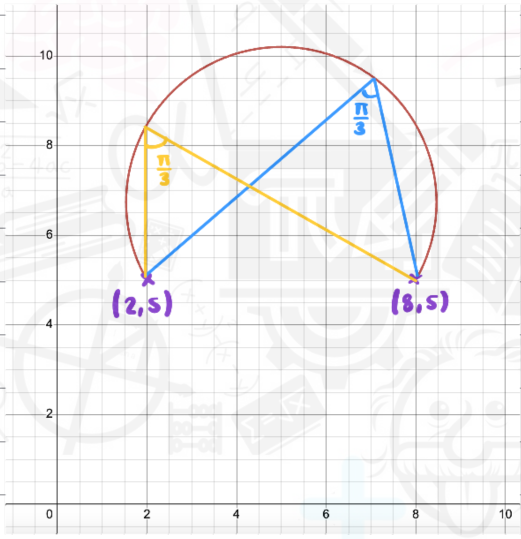
$a = 8+5i$

$b = 2+5i$

$\theta = \frac{\pi}{3}$

→ since $\frac{\pi}{3} < \frac{\pi}{2}$, the locus is the major arc of the circle

→ arc drawn anticlockwise from $8+5i$ to $2+5i$



b. as midpoint of (2,5) and (8,5)

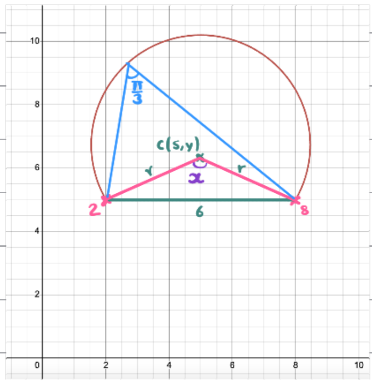
$\left(\frac{2+8}{2}, y\right) = \left(\frac{10}{2}, y\right) = (5, y)$

∴ centre of C has x-coord 5 //

P 6 5 5 0 6 A 0 2 2 3 2

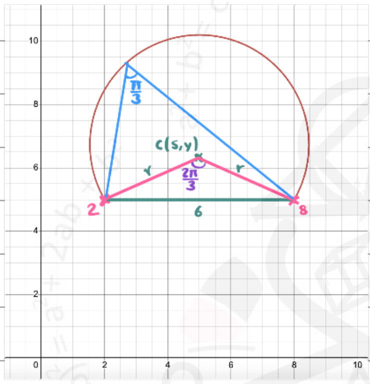
Question 8 continued

c.



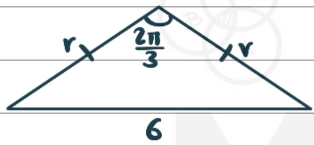
We need to find angle x

use circle theorem - angle at the centre is $2\times$ the angle of circumference

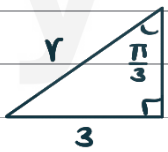


$x = 2 \times \frac{\pi}{3}$

$x = \frac{2\pi}{3}$



split triangle in half



$\sin\left(\frac{\pi}{3}\right) = \frac{3}{r}$

$r = 3 / \sin\left(\frac{\pi}{3}\right) = 2\sqrt{3}$

$\therefore$ radius $= 2\sqrt{3}$ //

(Total for Question 8 is 7 marks)

d. Circle eq^n can be written as:

$(x-5)^2+(y-y_1)^2=(2\sqrt{3})^2$

$(x-5)^2+(y-y_1)^2=12$

since we know $(2,5)$ lies on circle sub into circle eq^n and find y_1

$(2-5)^2+(5-y_1)^2=12$

$(-3)^2+(5-y_1)^2=12$

$9+y_1^2-10y_1+25=12$

$y_1^2-10y_1+22=0$

$$y_1=\frac{-(-10)\pm\sqrt{(-10)^2-(4\times1\times22)}}{2\times1}=5\pm\sqrt{3}$$

y-coord must be greater than 5, based off of graph (centre must lie in major arc)

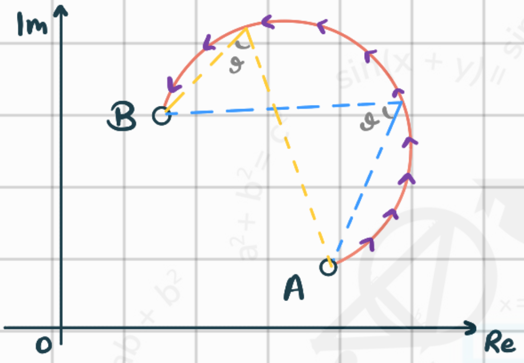
$\therefore$ reject $y=5-\sqrt{3}$

y-coord of centre of C: $5+\sqrt{3}$

The locus of points z that satisfy $\arg\left(\frac{z-a}{z-b}\right) = \theta$

where $\theta \in \mathbb{R}$, $\theta > 0$ and $a, b \in \mathbb{C}$, is an arc of a circle with endpoints A and B representing the complex no.s a and b, respectively.

The locus is the arc of a circle drawn anticlockwise from A to B.



$\arg\left(\frac{z-a}{z-b}\right) = \theta$

if $\theta < \frac{\pi}{2}$ then the locus is a major arc of a circle

if $\theta = \frac{\pi}{2}$ then the locus is a semi-circle.

if $\theta > \frac{\pi}{2}$ then the locus is a minor arc of a circle

9.

$$I_n = \int_0^{\frac{\pi}{2}} \sin^n 2x \, dx$$

(a) Prove that for $n \geqslant 2$

$$I_n = \frac{n-1}{n} I_{n-2}$$

(4)

(b) Hence determine the exact value of

$$\int_0^{\frac{\pi}{2}} 64 \sin^5 x \cos^5 x \, dx$$

(3)

a. $I_n = \int_0^{\frac{\pi}{2}} \sin^n 2x \, dx$

$I_n = \int_0^{\frac{\pi}{2}} \underbrace{\sin^{n-1} 2x}_{u} \; \underbrace{\sin 2x}_{v'} \, dx$

Integration by parts: $\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx$ Given in formulae booklet

$u = \sin^{n-1} 2x$ $\qquad v' = \sin 2x$

$u' = (n-1)\sin^{n-2} 2x \; 2\cos 2x$ $\qquad v = -\frac{1}{2}\cos 2x$

$I_n = \left[-\frac{1}{2}\cos 2x \sin^{n-1} 2x\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} - \cancel{\frac{1}{2}}\cos 2x \,(n-1)\sin^{n-2} 2x \; \cancel{2}\cos 2x \, dx$

factor out -1 since it is a constant

$I_n = \left[\left\{-\frac{1}{2}\cos\left(2\times\frac{\pi}{2}\right)\sin^{n-1}\left(2\times\frac{\pi}{2}\right)\right\} - \left\{-\frac{1}{2}\cos(2\times 0)\sin^{n-1}(2\times 0)\right\}\right] + \int_0^{\frac{\pi}{2}} \cos 2x\,(n-1)\sin^{n-2} 2x \cos 2x \, dx$

$I_n = \left[\{0\} - \{0\}\right] + \int_0^{\frac{\pi}{2}} \cos^2 2x\,(n-1)\sin^{n-2} 2x \, dx$

since it is a constant can factor out integral

use identity $\cos^2 x + \sin^2 x = 1$ - rearrange and remember it is $\cos^2(2x)$

$I_n = 0 + (n-1)\int_0^{\frac{\pi}{2}} (1 - \sin^2 2x)\sin^{n-2} 2x \, dx$

$I_n = (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2} 2x - \sin^n 2x \, dx$

split integral into 2

$I_n = (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2} 2x \, dx - (n-1)\int_0^{\frac{\pi}{2}} \sin^n 2x \, dx$

$I_n = (n-1) I_{n-2} - (n-1) I_n$

$\cancel{I_n} = n I_{n-2} - I_{n-2} - n I_n + \cancel{I_n}$

DO NOT WRITE IN THIS AREA

Question 9 continued

$nI_{n-2} - I_{n-2} = nI_n$

$I_{n-2}(n-1) = nI_n$) $\div n$ on both sides

$I_n = \frac{n-1}{n} I_{n-2}$

$I_n = \frac{n-1}{n} I_{n-2}$ //

b. $\int_0^{\frac{\pi}{2}} 64 \sin^5 x \cos^5 x \, dx = 64 \int_0^{\frac{\pi}{2}} \sin^5 x \cos^5 x \, dx$

take 64 out of integral

$= 64 \int_0^{\frac{\pi}{2}} (\sin x \cos x)^5 \, dx$

use double angle formulae

$= 64 \int_0^{\frac{\pi}{2}} (\frac{1}{2} \sin 2x)^5 \, dx$

$= 64 \int_0^{\frac{\pi}{2}} \frac{1}{32} \sin^5 2x \, dx$

take $\frac{1}{32}$ out of integral

$= 2 \int_0^{\frac{\pi}{2}} \sin^5 2x \, dx$

We are trying to find I_5, when $n=5$:

REMEMBER TO DO 2× AT END

$I_5 = \frac{5-1}{5} I_{5-2} = \frac{4}{5} I_3$

$I_3 = \frac{3-1}{3} I_{3-2} = \frac{2}{3} I_1$

$I_1 = \int_0^{\frac{\pi}{2}} \sin 2x \, dx = \left[-\frac{1}{2}\cos(2x) \right]_0^{\frac{\pi}{2}}$

$= \left[\left\{ -\frac{1}{2}\cos(2 \times \frac{\pi}{2}) \right\} - \left\{ -\frac{1}{2}\cos(2 \times 0) \right\} \right]$

$= \left[(\frac{1}{2}) - (-\frac{1}{2}) \right]$

$= 1$

Question 9 continued

$I_3 = \frac{2}{3}(1)$

$= \frac{2}{3}$

$I_5 = \frac{4}{5}\left(\frac{2}{3}\right)$

$= \frac{8}{15}$

remember, we must do $2 \times I_5$:

$2 \times \frac{8}{15} = \frac{16}{15}$

$\frac{16}{15}$ //

10.

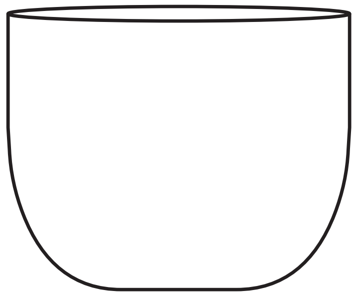


Figure 2

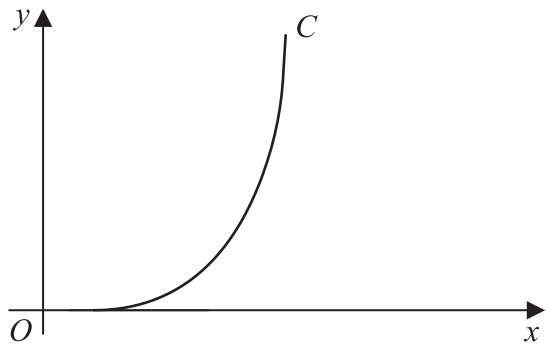


Figure 3

Figure 2 shows a picture of a plant pot.

The plant pot has

- a flat circular base of radius 10 cm
- a height of 15 cm

Figure 3 shows a sketch of the curve C with parametric equations

$$x = 10 + 15t - 5t^3 \qquad y = 15t^2 \qquad 0 \leqslant t \leqslant 1$$

domain given so use these limits

The curved inner surface of the plant pot is modelled by the surface of revolution formed by rotating curve C through 2π radians about the y-axis.

(a) Show that, according to the model, the area of the curved inner surface of the plant pot is given by

$$150\pi \int_0^1 \left(2 + 3t + 2t^2 + 2t^3 - t^5\right) \mathrm{d}t$$

(5)

(b) Determine, according to the model, the total area of the inner surface of the plant pot.

(4)

Each plant pot will be painted with one coat of paint, both inside and outside.
The paint in one tin will cover an area of 12 m^2

(c) Use the answer to part (b) to estimate how many plant pots can be painted using one tin of paint.

(2)

(d) Give a reason why the model might not give an accurate answer to part (c).

(1)

a. $S_x = 2\pi \int y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$ GIVEN IN FORMULAE BOOKLET

HOWEVER Q IS ASKING TO REVOLVE AROUND Y-AXIS SO IN FORMULAE REPLACE Y WITH X IN FORMULAE

Question 10 continued

$x = 10 + 15t - 5t^3$

$\frac{dx}{dt} = 15 - 15t^2$

$\left(\frac{dx}{dt}\right)^2 = (15 - 15t^2)^2 = 225t^4 - 450t^2 + 225$

$y = 15t^2$

$\frac{dy}{dt} = 30t$

$\left(\frac{dy}{dt}\right)^2 = (30t)^2 = 900t^2$

x used instead of y

$S_x = 2\pi \int_0^1 (10 + 15t - 5t^3)\sqrt{225t^4 - 450t^2 + 225 + 900t^2}\, dt$

$S_x = 2\pi \int_0^1 (10 + 15t - 5t^3)\sqrt{225t^4 + 450t^2 + 225}\, dt$

factor out 225

$S_x = 2\pi \int_0^1 (10 + 15t - 5t^3)\sqrt{225(t^4 + 2t^2 + 1)}\, dt$

$S_x = 2\pi \int_0^1 (10 + 15t - 5t^3)\sqrt{225(t^2 + 1)^2}\, dt$

split surd into 2

$S_x = 2\pi \int_0^1 (10 + 15t - 5t^3)\sqrt{225}\sqrt{(t^2 + 1)^2}\, dt$

$S_x = 2\pi \int_0^1 (10 + 15t - 5t^3)\, 15(t^2 + 1)\, dt$

factor 15 out of integral

$S_x = 30\pi \int_0^1 (10 + 15t - 5t^3)(t^2 + 1)\, dt$

$S_x = 30\pi \int_0^1 10t^2 + 15t^3 - 5t^5 + 10 + 15t - 5t^3\, dt$

$S_x = 30\pi \int_0^1 5(2 + 3t + 2t^2 + 2t^3 - t^5)\, dt$

factor 5 out of integral

$S_x = 150\pi \int_0^1 2 + 3t + 2t^2 + 2t^3 - t^5\, dt$ //

b. $S_x = 150\pi \left[2t + \frac{3t^2}{2} + \frac{2t^3}{3} + \frac{2t^4}{4} - \frac{t^6}{6}\right]_0^1$

$S_x = 150\pi \left[\left\{2(1) + \frac{3(1)^2}{2} + \frac{2(1)^3}{3} + \frac{2(1)^4}{4} - \frac{(1)^6}{6}\right\} - \left\{2(0) + \frac{3(0)^2}{2} + \frac{2(0)^3}{3} + \frac{2(0)^4}{4} - \frac{(0)^6}{6}\right\}\right]$

$S_x = 150\pi\left[\frac{9}{2}\right] = 675\pi$

Question 10 continued

We also need area of circular base:
"Q states flat circular base has radius 10cm.

$\therefore$ area of circle: $\pi r^2 = \pi(10)^2 = 100\pi$

$675\pi + 100\pi$
$= 775\pi \text{ cm}^2$

$775\pi \text{ cm}^2$ //

c. Since plant pot will be painted inside and outside do: $775\pi \times 2 = 1550\pi \text{ cm}^2$

change from $\text{cm}^2 \to \text{m}^2$

$10000 \text{ cm}^2 = 1 \text{m}^2$
$1550\pi \text{ cm}^2 = 0.155 \text{m}^2$

$\frac{12}{0.155} = 24.64334603$

$\therefore$ 24 plant pots can be painted with 1 tin of paint //

d. The surface area outside of the plant pot will be more than the inside //